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The question of Heisenberg uncertainty principle: Inevitable alternative interpretations and applications

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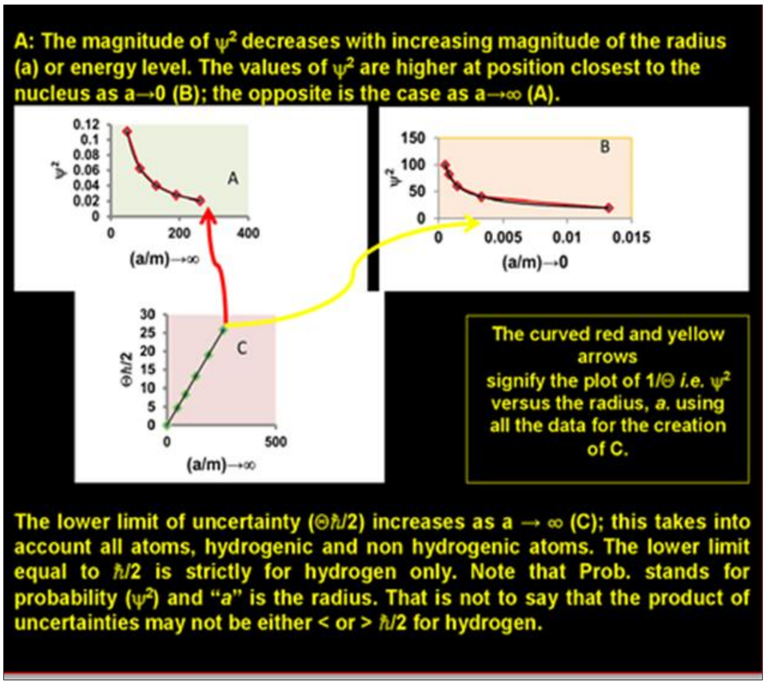
Abstract

With hydrogen apart from other elements, fractions and multiples of $\hbar/2$, 0.0053–0.0264 corresponding to the radii ranging between $a_0/100$ and $a_0/20$ (a_0 is the Bohr's radius), and 5-26 corresponding to the radii ranging between $4a_0$ and $49a_0$ were demonstrated; the probability of locating the position (a) and momentum (p) increases with decreasing radii. The equations for the determination of the fractions and multiples of $\hbar/2$ were derived. Computation gave values less than and higher than $\hbar/2$; the latter was found to be restricted to hydrogen atom only. A relation between uncertainties and pa is for future study.

Keywords: Average ionization energies, Heisenberg uncertainty principle, hydrogen and other selected elements, reduced Planck's constant, Rydberg constant.

Graphical abstract


A: The magnitude of ψ^2 decreases with increasing magnitude of the radius (a) or energy level. The values of ψ^2 are higher at position closest to the nucleus as a→0 (B); the opposite is the case as a→∞ (A).

The curved red and yellow arrows signify the plot of 1/Θ *i.e.* ψ^2 versus the radius, *a*, using all the data for the creation of C.

The lower limit of uncertainty (Θħ/2) increases as a → ∞ (C); this takes into account all atoms, hydrogenic and non hydrogenic atoms. The lower limit equal to ħ/2 is strictly for hydrogen only. Note that Prob. stands for probability (ψ^2) and "a" is the radius. That is not to say that the product of uncertainties may not be either < or > ħ/2 for hydrogen.

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1. Introduction

So much altercation for and against the Heisenberg uncertainty principle (HUP) abounds in the literature. A recent preprint report shows that HUP can be proved. [1]. There is also a post-doctoral article on the mathematical aspect of HUP [2]. The complexity is extremely high and comprehensible only to those with the highest level of all kinds of mathematics, a typical example that provoked Mills' [3] opposition against the view that "the electron is a point with no volume with a vague probability wave requiring that the electron have multiple positions and energies, including negative and infinite energies simultaneously. It is important to add that it is not just a 75-year issue because, as noted elsewhere [4], Abu Bakr Muhammade bin Zakariyya (*Wikipedia*), who may have been around for more than 1000 years, has a view about the atom that is similar to the objected 20th century notion about the atom. It is noteworthy that most research regarding HUP has violated the concept of limits in the magnitude of uncertainties [5]. Hence, the question of HUP with Planck's constant, as opposed to recent formulations where Planck's constant seems to be masked, is to be answered with alterative interpretations with likely applications with the aim of showing that in its original form, HUP is related to energy functions. The research begins bearing in mind what is regarded as vague [6] but also disputed in this research with the following objectives: 1) To undertake the derivation of an equation unifying the Heisenberg uncertainty principle and atomic principles; 2) give evidence that half of the reduced Planck's constant is strictly for hydrogen.

2. Theory

In this section, a theoretical overview of views in the literature regarding HUP is undertaken in addition to the most important aspect, the derivation of equations that make Planck's constant indispensible; this is in addition to the derived equation of Planck's constant, which has a place for the Rydberg constant. Further derivation is applicable to multi-electron atoms.

2.1. Overview

The HUP [7] is said to describe a trade-off between the error of a measurement of one observable and the disturbance caused on another complementary observable, such that their product should not be less than the limit set by Planck's constant. It has been argued that Ozawa's 1988 model of position measurement breaks Heisenberg's relation and reveals a 2003 [8] alternative relation for error and disturbance that was proved to be universally valid [9]. It would appear from this statement that Planck's constant is no more relevant. Before this, let us recall earlier equations related to HUP. It has been observed that Kennard's work [10] revealed that Heisenberg was credited with a constraint for state preparation given as the product of standard deviation (SD), such as:

$$\varepsilon(q') \, \varepsilon(p') \sim h, \tag{1}$$

where $\varepsilon(q')$ is the "mean error" of position measurement and $\varepsilon(p')$ is the mean error of the simultaneous momentum measurement, which caused a "discontinuous change" of the momentum; h is the usual Planck's constant. This follows Heisenberg's derivation of what was referred to as the "standard deviation for Gaussian wave functions, later called minimum uncertainty wave packets." [11].

$$\sigma(q') \, \sigma(p') = \hbar/2, \tag{2}$$

Next is the inequality credited to Kennard [10] and given as:

$$\sigma(q') \, \sigma(p') \geq \hbar/2, \tag{3}$$

By the inequality (In-Eq. (3)) the lower bound of the relation (Eq. (1)) is set as:

$$\varepsilon(q') \, \varepsilon(p') \geq \hbar/2, \tag{4}$$

Meanwhile, Eq. (3) in its rewritten form [12] is:

$$\sigma(q) \, \sigma(p) \geq \hbar/2, \tag{5}$$

where respectively, the standard deviations σ (q) and σ (p) of the position q and the momentum p are defined, for instance, by $\sigma\langle q^2\rangle - \sigma\langle q\rangle^2$ and $\sigma\langle p^2\rangle - \sigma\langle p\rangle^2$ and $\langle ... \rangle$ denotes mean value in a given state. Then, is the 1929 generalized form of Eq. (3) given by Robertson [12]:

$$\sigma(A)\,\sigma(B) \geq |\langle [A, B] \rangle|/2 \tag{6}$$

where A and B are arbitrary pair of observables and [A, B] = AB−BA. Then comes the arbitrary form of Heisenberg's relation given as:

$$\varepsilon\,(A)\,\eta(B) \geq |\langle [A, B] \rangle|/2 \tag{7}$$

Equation (7) is regarded as having been accepted to hold for the error $\varepsilon\,(A)$ of any A-measurement and the disturbance η (B) caused by that measurement on an observable B; the relation is observed to have been proven only in limited circumstances [8,13,14].

The question of joint measurements seems slightly intelligible because, with an appropriate technology, one can simultaneously measure the position and momentum of an electron without perturbation and associated dislocation, not only in terms of a change in position but also in terms of a change in momentum. Therefore, describing the following view (one of three views) as vague [6] seems inappropriate. It is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized.

If twenty "measurements" were taken for the determination of the first ionization energies of any atom in its ground state, it is not unlikely that some of the values may be > the average while some are less than the average; the likelihood of very few being equal to the average value cannot be ruled out entirely. Those values that are above average show that the electron may be closer to the reference point, the atomic nucleus, contrary to those that are lower than average. These are expressions of position relative to the nucleus, and from the same measurement, the momentum can also be clearly quantified. This looks like joint measurements of position and momentum, which according to Busch [6] are in fact possible with arbitrary accuracy, but yet it is like using one "primitive catapult-powered stone to kill two birds". Now here comes Ozawa's alternative relation for error and disturbance that was theoretically proved to be universally valid: According to the author, any measurement of an observable A in a state $|\psi\rangle$ (a layman like the author of this unacclaimed paper may wish to know the kind of state) with an error $\varepsilon(A)$ causes the disturbance $\eta(B)$ on another observable *B and* satisfies the equation:

$$\varepsilon(A)\eta(B) + \varepsilon(A)\sigma(B) + \sigma(A)\eta(B) \geq |\langle [A, B] \rangle|/2 \tag{8}$$

where $\sigma(A)$ and $\sigma(B)$ stand for the SD in the state $|\psi\rangle$ while the error-disturbance product, $\varepsilon(A)\eta(B)$, is $\ll$ the lower bound of Eq. (7). It was found that the outcome of the experiment conducted to evaluate Eq. (8) validated Ozawa's equation while the Heisenberg equation broke down, or rather, was violated [9]. The error $\varepsilon(A)$ is the root mean square of the difference between the "meter observable M" after the interaction and the "meter observable A" before the interaction. The disturbance $\eta(B)$ is the root-mean-square of the change in the observable B during the measuring interaction. These definitions of error and disturbance are taken to be generalizations of their classical definitions [9]. These definitions are despite the equations $\varepsilon(A{=}Z) = 2\sin\theta$ and $\eta(B{=}X) = 2\sin((\pi/2)-\theta)$ expressing $\varepsilon(A{=}Z)$ and $\eta(B{=}X)$, respectively. In trigonometry, the sine, cosine, tangent, cotangent, cosine, secant, *etc.* are ratios that cannot be translated into units of energy given by $h\varepsilon(f)$ where $\varepsilon(f)$ is the error in measured frequency. Alternatively, one can assume the relation $h\sigma(f)$ where $\sigma(f)$ is the uncertainty in the measured frequency. Besides, in statistics, uncertainty and error (the standard error of the mean in particular) are not the same, even if ordinary error may coincidentally be equal to uncertainty. Following the formalism given as [9]:

$$\eta(p) = \langle p^2 \rangle^{1/2} \geq \sigma(p) \tag{9}$$

where $\langle \ldots \rangle$ is the post measurement mean value [9] and the counterpart,

$$\eta(q) = \langle q^2 \rangle^{1/2} \geq \sigma(q), \tag{10}$$

One can also imagine an electron that is in motion within the bounds of the atom under the influence of the nucleus, with several positions whose distance from the nucleus may vary but not enough to escape the nuclear influence. Besides, Eqs (9) and (10) give the impression that $\sigma(p)$ and $\sigma(q)$ are respectively, natural consequences of the disturbance represented by $\eta(p)$ and $\eta(q)$. However, a question that needs to be answered is whether the mean can ever be equal to uncertainty, let alone be greater. It may be necessary to focus on the consequences. This scenario is relevant to an electron that has been perturbed.

2.2. Background derivations

Taken as an atom, a round three-dimensional space gives an area of a sphere given as $4\pi a^2$ (where *'a'* is the average ionization potential-dependent radius of any atom other than hydrogen). Whatever distance between the electron and the nucleus in the course of an experimental measurement of its ionization potential, one can consider different values of "a," including the Bohr's radius (a_0 exclusively reserved for hydrogen), such that:

$$\frac{\eta(a^2)}{\tau} m_e = \frac{4\pi\langle a^4\rangle^{½}}{\tau} m_e \geq \frac{4\pi}{\tau} m_e \sigma(a^2) \tag{11}$$

where τ may be taken as the duration of measurement.

Equation (11) or its second aspect, the In-Eq. (11) (*note that In-Eq. stands for inequality or not equal*) can be related to the Planck's constant (reduced Planck's constant) just as the energy of a randomly moving molecule of a gas can be related to the same constant as follows: $\lambda_{dB} = \Theta h/m_g v_g$ where λ_{dB}, m_{g}, v_{g} and Θ are the de Broglie wave length, average molecular mass of any gas, speed, and randomness factor, an integer that is $\gg 1$[15]. A dimensionless quantity cannot be appropriately compared with a dimensional quantity, contrary to the suggestion that where HUP failed (became violated or flawed), Ozawa was successful or valid as guided by Eq. (8). The error aspect of Eq. (11) is then,

$$\frac{\eta(a^2)}{\tau} m_e = \frac{4\pi\langle a^4\rangle^{½}}{\tau} m_e \geq \frac{4\pi}{\tau} m_e \varepsilon(a^2)/\Phi_{\varepsilon\to a} \tag{12}$$

In Eq. (12), $\Phi_{\varepsilon\to a}$ is the error coefficient (it may either be < or > 1; a ratio of any reference value if known or the mean to the observed as applicable to the radius) whose magnitude would depend on a_{i}. Its relevance can be appreciated in Eq. (13) below, which points to the fact that error and uncertainty are not the same. Other variants of Eq. (11) can be derived as follows: Bearing in mind the original Heisenberg inequality (HIE), it would appear that:

$$\frac{4\pi}{\tau} m_e \varepsilon(a^2) = \Phi_{\varepsilon\to a}\hbar/2 \tag{13}$$

$$\frac{4\pi}{\tau} m_e \sigma(a^2) = \hbar/2 \tag{14}$$

Meanwhile, as in a preprint [16], a_{i} is related to wavenumber according to the following equations:

$$\nabla = n a_0 (2E_{\mathrm{H}}^3 m_e)^{½}/h^2 R_\infty c\, a_i, \tag{15}$$

$$\nabla = n/2\, \pi a_i, \tag{16}$$

Therefore, error and uncertainty dovetail into error and uncertainty in the wavenumber, which is equivalent to error and uncertainty in momentum. Hence, one can rewrite Eqs (13) and (14) as follows:

$$\frac{n^2}{\tau\,\pi} m_e \varepsilon(\nabla^{-2}) = \Phi_{\varepsilon}\hbar \tag{17}$$

$$\frac{n^2}{\tau\,\pi} m_e \sigma(\nabla^{-2}) = \hbar \tag{18}$$

Recall the equation of a wavenumber given as $1/\lambda_{\mathrm{e}} = m_{\mathrm{e}} v_{\mathrm{e}}/h$ ($/\lambda_{\mathrm{e}}$ and v_{e} are the de Broglie wave length of the electron and the speed of the electron, respectively; for the sake of simplicity, $1/\lambda_{\mathrm{e}} = p/h$)); substituting p/h into Eqs (17) and (18) gives, after rearrangement, the equation of error and uncertainty in the reciprocal of the momentum or the speed given that the rest mass or, if preferred, the reduced mass of the electron is constant.

$$\varepsilon(p^{-2}) = \Phi_{\varepsilon\to p}\, \tau/2n^2\, m_e h, \tag{19}$$

In terms of speed, Eq. (19) becomes:

$$\varepsilon(v_e^{-2}) = \Phi_{\varepsilon\to v_e}\tau\, m_e/2n^2\, h, \tag{20}$$

$$\sigma(p^{-2}) = \tau/2n^2\, m_e h, \tag{21}$$

$$\sigma(v_e^{-2}) = \tau\, m_e/2n^2\, h, \tag{22}$$

In the absence of a reference, the error may be taken as the difference between the measured momentum or speed and the corresponding arithmetic mean. The standard deviation needs input from all the errors for its determination. However, the question that can be answered is: "What is the possibility that the inverse of Eqs (19) to (22) leads to?"

$$\varepsilon(p^2) = 2n^2 m_e h/\Phi_{\varepsilon\to p}\, \tau, \tag{23}$$

where $\Phi_{\varepsilon\to p}$ is the error coefficient applicable to p and,

$$\varepsilon(v_e^2) = 2n^2 h/\Phi_{\varepsilon\to v_e}\, \tau\, m_e, \tag{24}$$

where $\Phi_{\varepsilon\to v_e}$ is the error coefficient applicable to v_e.

$$\sigma(p^2) = 2n^2 m_e h/\tau, \tag{25}$$

$$\sigma(v_e^2) = 2n^2 h/\tau\; m_e, \tag{26}$$

In statistics, population is very important, according to which either one adopts parametric or nonparametric methods for the determination of standard deviation. Adopting the parametric method as a preferred option demands that the number of determinations be > 6; representing the number of determinations as, ϕ one can take the root of the mean of the errors and uncertainties in the squared variables as follows:

$$\left(\frac{\sum_1^{\phi} \varepsilon(p^2)}{\phi}\right)^{½} = \left(\frac{2\, n^2 m_e h}{\tau}\right)^{½} \Phi_{\varepsilon\to p}^{-½}, \tag{27a}$$

$$\left(\frac{\sum_1^{\phi} \varepsilon(p^2)}{2\,\phi}\right)^{½} = \left(\frac{n^2 m_e h}{\tau}\right)^{½} \Phi_{\varepsilon\to p}^{-½}\ , \tag{27b}$$

$$\left(\frac{\sum_1^{\phi} \varepsilon(p^2)}{2\,\phi}\right)^{½} \Phi_{\varepsilon\to p}^{+½} = \left(\sigma(p^2)\right)^{½} = \left(\frac{n^2 m_e h}{\tau}\right)^{½}, \tag{28}$$

The same procedure applies to v_e, which is set aside in preference to p for now. Meanwhile, a similar procedure should be applied to Eqs (13) and (14) to obtain the following:

$$\left(\frac{\sum_1^{\phi} \varepsilon(a^2)}{\phi\,\pi}\right)^{½} = \left(\frac{\Phi_{\varepsilon\to \mathrm{a}}\, h\,\tau}{4\,\pi^2 m_e}\right)^{½}, \tag{29}$$

$$\left(\frac{\sum_1^{\phi} \varepsilon(a^2)}{\phi\,\pi}\right)^{½} (\Phi_{\varepsilon\to \mathrm{a}})^{-½} = \left(\sigma(a^2)\right)^{½} = \left(\frac{h\,\tau}{4\,\pi^2 m_e}\right)^{½}, \tag{30}$$

Now, one can invoke “Heisenberg energy principle” (HEP) and HUP in order to derive the following equations: Beginning with Eqs (28) and (30), one gets:

$$\left(\frac{\sum_1^{\phi} \varepsilon(p^2)}{2\phi}\right)^{½} \Phi_{\varepsilon\to p}^{+½} \left(\frac{\sum_1^{\phi} \varepsilon(a^2)}{\phi\,\pi}\right)^{½} (\Phi_{\varepsilon\to \mathrm{a}})^{-½} = \left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½} = \left(\frac{n^2 m_e h}{\tau}\right)^{½} \left(\frac{h\,\tau}{4\,\pi^2 m_e}\right)^{½}, \tag{31}$$

The position in this research is to remain closely as a student (rather than high-ranking science scholars imbued with "intimidating" postdoctoral mathematics) of Heisenberg and Bohr, on account of which one bears in mind the rather general expression, $\Delta x\Delta p \geq \hbar/2$, and Eqs (15) and (16), whose combination as in the literature can give a variant of the equation of the Rydberg constant, and in this research the equation of Planck’s constant in which Bohr’s radius plays a key role. Therefore, Eq. (31) can be rewritten in simpler forms to give:

$$\Phi_{\varepsilon\to \mathrm{P}}^{½} \left[\langle\left(\varepsilon(p^2)\right)^{½}\rangle \langle\left(\varepsilon(a^2)\right)^{½}\rangle\right]/(8\,\pi\,\Phi_{\varepsilon\to a})^{½} = n\hbar/2, \tag{34}$$

$$\left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½} = n\hbar/2, \tag{35a}$$

Two clear statements, that the experimentally generated data clearly demonstrate that, Ozawa's relation is always valid, whereas Heisenberg's relation is false for all measurement strengths and the confirmation that the ''three-state-method'' successfully determines the error and the disturbance of the photonic measuring apparatus [9] seem to lend credence to Hill's view [3]; such view is that the HUP is wrongly interpreted as: the uncertainty in the measured momentum times the uncertainty in the measured position must be no less than $\hbar$ as given by $\Delta x \Delta p \geq \hbar/2$; yet the same author seems to agree that the HUP gives a lower limit to the product of the uncertainty in the momentum and the uncertainty in the position—not the product of the momentum and the position; the HUP is the mathematical expression for the statistical error in the variables of the wavefunction such as those assigned to the position and momentum of the electron. This seems to go with the issue of error in measurement in Baek *et al.'s* work [9], but the inequality has no place for Planck's constant in the enabling mathematical expression.

Similar to the experimental study [8] containing few mathematical equations rich in set theory and an all-out mathematical exposition of different kinds, including much about set theories, is the works of Busch [6] and Yang *et al.* [2], which seem to give credence to Mill's [3] claims that despite its successes, after decades of futility, quantum mechanics and the intrinsic HUP have not yielded a unified theory, are still purely mathematical, and have yet to be shown to be based in reality. It is not all about the experimentalist, Ozawa, but similar experimentally based opinions have been held for years. The "contractive states," the product of the free evolution of a subclass of photon coherent states, caused a narrowing of the position uncertainty in breach of the standard quantum limit [17]. The conventional bounds to the precision of measurements, such as the shot noise limit or the standard quantum limit, are regarded as not being as fundamental as the Heisenberg limit, and as such, they can be beaten using quantum strategies that explore "quantum tricks" such as squeezing and entanglement [18]. *The perspectives of high-ranking specialists are not clearly understood by others.* Exploring a precision position measuring apparatus has put into question the HUP, or better yet, violated the HUP [19].

3. Materials and Methods

The research is theoretical and computational. However, the key data for evaluating the derived equations where appropriate were experimental data found in common standard text books and on the internet encyclopedia.

4. Results and discussion

This section begins with the understanding that error and uncertainty are not the same, such that this and any other views in the literature demand reinterpretations. Currently, there is the issue of the violation of HUP; this would have been unnecessary if a rigid adherence to the undefined upper limit and the possibility of equality of the product of uncertainties (PUCs) and half of the reduced Planck's constant were not the case. In a situation where $\sigma(p) = [\hbar/2]/\sigma(q)$, $\varepsilon(p) = \phi_P\hbar/2\varepsilon(q)$ and $4\pi m_e\ [\varepsilon(a^2)]^{½}/\tau = [\phi_{\varepsilon\rightarrow a}\hbar/2]/[\varepsilon^2)]^{½}$, as implied in Eqs (9), (10) and (11) respectively, the uncertainties must be between $\ll$ exp. (−17) and $\gg$ exp. (−17) and this includes the relation, $4\pi m_e\ [\sigma(a^2)]^{½}/\tau = [\hbar/2]/[\sigma(p^2)]^{½}$ where in particular a falls within the sphere of influence of the nucleus and τ must be $\ll 1$. Where $4\pi m_e\ [\sigma(a^2)]^{½}/\tau \geq [\hbar/2]/[\sigma(p^2)]^{½}$ the uncertainties must be very high where in particular $a \rightarrow \infty$ even if p is low. Leveraging on these pieces of information, one can consider three classes of perturbation (disturbance), *viz*., lower energy level oriented perturbation (LOP), partial ionization (higher energy level) oriented perturbation (HOP), and total ionization oriented perturbation (TOP); the role of energy level has become evident in that if an electron remains in an unspecified point in space at a particular energy level, the equality in Eqs (34) and (35a) stands if $\sigma(p)$ and $\sigma(q)$ are respectively, $\ll$ exp. (−17) and $\gg$ exp. (−17) and *vice versa*.

If, on the other hand, the errors are infinitesimal, the PUCs should be $\ll \hbar/2$. With electron capture, there is no question of where the electron is; there cannot be measurement error, unlike when the location of an electron is at an infinite distance from the nucleus. The momentum cannot be equal to zero, though it may be low. However, as long as a 3-dimensional space is available for an electron at any energy level, while it may not escape the attraction of the nucleus, it cannot be assumed to be a planner in its region of motion. Thus, the product of the uncertainties may also be $\ll \hbar/2$. Therefore, there are two possibilities. Equation (35b) illustrates this issue, and it may be very illustrative of what happens in metals, specifically the transition elements in which the valence shell electrons are relatively free. PUCs of position from nucleus and momentum could be very high and greater than $\hbar/2$; PUCs may tend to $\hbar/2$ if not $\approx \hbar/2$.

$$\left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½} \geq n\hbar/2, \tag{35b}$$

If, for whatever reason, an electron ascends to a higher energy level while losing momentum, the $\Delta x \Delta p$ is very likely to be > $\hbar/2$. Not just for the fancy of it, Eq. (35a) or (35b) can be rewritten as:

$$\left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½}/(n+i) > \hbar/2, \tag{36}$$

where i is any integer such that $n+i$ is > 0 and ≥1 depending on the value of n. At a lower energy level, momentum is high, so any measurement in that regard may likely yield a small uncertainty, instrumental error notwithstanding. If the electron moves to a lower energy level, $\Delta x \Delta p$ may be < $\hbar/2$; there is a gain in momentum because the distance from the nucleus has decreased. Then the inequality takes the form:

$$\left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½}/(n-i) < \hbar/2, \tag{37}$$

Again, i is such that $(n-i)$ is < n_{n} (the highest energy level for the atom in its ground state). Note that if all measurements are error-free, the product of zero and either of the two measurements gives zero, which is < $\hbar/2$. So far, energy has been introduced into HUP theory, but what needs to be understood is that a qualitative analysis of the necessary equations and inequalities has been carried out. Besides, $\hbar/2$ is the middle (or better still, the median) limit of uncertainties, as opposed to being the lower limit referred to in the literature [3]. This is clarified by the following derivations:

It is not all about violation; concern for principle should prevail. There is no doubt that there is a theoretical and even a practical limit, such as that defined when ionization takes place: where does one locate an electron that has been expelled from nuclear influence? When an electron is expelled, the energy level is undefined (infinite), and it is designated as n_∞; thus,

$$\left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½}/n_\infty > 0 > \hbar/2, \tag{38}$$

Note that as $n \to \infty$, the speed of the electron outside the nuclear influence does not → ∞; it may tend to zero outside any electrostatic influence. So far in this analysis and discussion, the value of the initial n has not been indicated. Here, one can state preemptively that n belongs to hydrogenic atoms and ions. The ground for this assertion is as follows: Meanwhile, from Eqs (15) and (16), as in a preprint [16], the Planck's constant is given as:

$$h = (8m_e E_{\mathrm{H}}^3)^{¼}\left(\frac{\pi a_0}{R_\infty c}\right)^{½}, \tag{39}$$

Then, the reduced Planck's constant is given as:

$$\hbar = (m_e E_{\mathrm{H}}^3/2)^{¼}\left(\frac{a_0}{\pi R_\infty c}\right)^{½}, \tag{40}$$

It is instructive to state that, based on Bohr's equation for the average ionization energy of hydrogen, h, is also given as: $h = (e^2/\varepsilon_0)(m_e/8\,E_{\mathrm{H}})^{½}$. The difference is, of course, the absence of R_∞ in the latter. The interest in Eq. (40) lies in showing that Bohr's principle expressed in the average ionization energy of hydrogen and Bohr's radius and partly with the Rydberg constant, R_∞, has a relationship with HUP as follows:

$$\left(\sigma(p^2)\right)^{½}\left(\sigma(a^2)\right)^{½} \geq n(m_e E_{\mathrm{H}}^3/2)^{¼}\left(\frac{a_0}{\pi R_\infty c}\right)^{½}/2, \tag{41}$$

Based on the Planck constant-free equation [4], for the determination of the radius of any atom, Eq. (41) can be written as follows: First, the Bohr's radius is given as: $a_0 = e^2/8\,E_{\mathrm{H}}\,\pi\,\varepsilon_0$ which has always been because Z_{eff} is = 1 (this is unlike multi-electron atoms). Then substitute into Eq. (41) to obtain the following, after rearrangement:

$$\geq (n^3)^{½}\frac{e}{2\pi}\left(\frac{1}{\varepsilon_0 R_\infty c}\right)^{½}\left(\frac{E_{\mathrm{H}} m_{\mathrm{e}}}{128}\right)^{¼}, \tag{42}$$

Note that the right-hand side of In-Eq. (42) is = $(n^3)^{½}\,\hbar/2$. However, the value of n is one because Eq. (42) is primarily for hydrogenic ions or atoms. The version for a multi-electron or multi-proton hydrogenic atom, where n is ≥ 1, is derived as follows: For the later, the equation is: $a_{\mathrm{i}} = e^2 n/8\,(E_{\mathrm{i}}\,E_{\mathrm{H}})^{½}\,\pi\varepsilon_0$; a generalization of Eq. (41) means that the radii of multi-electron atoms and ions are relevant such that:

$$\left(\sigma(P^2)\right)^{½}\left(\sigma(a^2)\right)^{½} \geq (n^3)^{½}\frac{e}{2\pi}\left(\frac{1}{\varepsilon_0 R_\infty c}\right)^{½}\left(\frac{E_{\mathrm{H}}^2 m_{\mathrm{e}}}{128 E_{\mathrm{i}}}\right)^{¼}, \tag{43}$$

Although Eq. (43) specifies multielectron atoms, it is nevertheless a general one as long as n is equal to one (and E_i is equal to E_H for hydrogen) for hydrogenic atoms and ions. The point is clear considering one-proton hydrogenic atoms, multiproton hydrogenic ions, and multielectron atoms. It can be clearer if one rearranges Eq. (43) in a way that shows that different atoms should possess different multiples of half of reduced Planck's constant, as follows:

$$\geq \left(\frac{n^6}{E_\mathrm{i}}\right)^{1/4} \frac{e}{2\pi} \left(\frac{1}{\varepsilon_0 R_\infty c}\right)^{1/2} \left(\frac{E_\mathrm{H}^2 m_\mathrm{e}}{128}\right)^{1/4}, \tag{44}$$

All parameters in Eq. (44) are fundamental constant except n and E_i.

As shown in Table 1, one-proton hydrogenic atoms, a single energy level multielectron atom such as helium, and all multiproton hydrogenic ions show values that are $< \hbar/2$: This means that the uncertainties are likely to be very low. Precision in measurement is one aspect most likely where the electron (s) is strongly under the influence of the nucleus. Far away from the nucleus, precision measurement could be very difficult, if not impossible. Table 1 shows quantitative evidence with large mass-number atoms. Besides, the enabling (in) equalities are given as:

$$\geq \frac{n^2_{\rightarrow\infty}\, e}{2\pi} \left(\frac{E_\mathrm{H}}{\varepsilon_0 R_\infty\, c\, n_i}\right)^{1/2} \left(\frac{m_\mathrm{e}}{128 E_\mathrm{i}}\right)^{1/4}, \tag{45}$$

where $n_{\rightarrow\infty}$, n_i and E_i are energy level towards infinity, initial n^{th} energy level and associated ionization energy for this purpose. The model concerns positions less than infinity within the sphere of highly attenuated influence of the nucleus, as in metals with free valence orbital electrons; distances much longer than the nucleus-valence shell distance are also envisaged where the attractive kinetic energy is very low. To preclude any doubt (not for the sake of unimpressive numbers), other (in) equalities are:

$$\leq \frac{n^2_{\rightarrow\infty}\, e}{2\pi} \left(\frac{E_\mathrm{H}}{\varepsilon_0 R_\infty\, c\, n_i}\right)^{1/2} \left(\frac{m_\mathrm{e}}{128 E_\mathrm{i}}\right)^{1/4}, \tag{46}$$

$$\approx \frac{n^2_{\rightarrow\infty}\, e}{2\pi} \left(\frac{E_\mathrm{H}}{\varepsilon_0 R_\infty\, c\, n_i}\right)^{1/2} \left(\frac{m_\mathrm{e}}{128 E_\mathrm{i}}\right)^{1/4}, \tag{47}$$

Reviewing Table 1, one sees clearly that different elements differ in the magnitude of the minimum product of uncertainties because of their differences in ionization energies, including those of the same period. The values range between ≈ 0.9 and 4 times half of the reduced Planck's constant with the 1st ionization energy; they are between ≈ 0.7 and 3 times half of the reduced Planck's constant with the 2nd ionization energy for some elements, H and He through F. Between 7th and 8th pqn, the minimum product of uncertainties is between 42 and 57 times half of the reduced Planck's constant. Among the period III elements, the minimum product of uncertainties ranges between ≈ 3 and 7 times half the reduced Planck's constant with the 1st ionization energy, while it is between ≈ 2 and 5 with the 2nd ionization energy (Ne through Ar). Between 8th and 9th pqn, it is ≈ between 40 and 57 times half the reduced Planck's constant. All the values of the minimum product of uncertainties for the heavier elements (e.g., I, Ba, Mo, etc.) were correspondingly > the values for their lighter-weight counterparts. All the values for positions where the Z^{th} ionization is the case were less than the minimum product of uncertainties (0.527 exp. (−34) J s), which is strictly for hydrogen, excluding any other hydrogenic ion because of the higher atomic number, Z.

Table 1 Fractions and multiples of half of the reduced Planck's constant (the minimum product of uncertainties) at different ionization energies

Period II atoms	**H**	**He**	**Li**	**Be**	**B**	**C**	**N**	**O**	**F**
1st Θ $\hbar/2$ (Js)/exp.(−34)	0.527	0.455	1.879	1.638	1.687	1.563	1.466	1.490	1.401
2nd Θ $\hbar/2$ (Js)/exp.(−34)	-	0.373	0.343	1.386	1.278	1.227	1.176	1.177	1.131
(n+6)th Θ $\hbar/2$ (Js)/exp.(−34)	25.825	22.272	30.060	26.210	26.991	25.040	23.455	23.842	22.418
Zth Θ $\hbar/2$ (Js)/exp.(−34)	-	0.373	0.3043	0.264	0.236	0.215	0.199	0.186	0.176
Period III atoms	Ne	Na	Mg	Al	Si	P	S	Cl	Ar
1st Θ $\hbar/2$ (Js)/exp.(−34)	1.328	3.493	3.163	3.364	3.112	2.889	2.931	2.770	2.640
2nd Θ $\hbar/2$ (Js)/exp.(−34)	1.131	1.092	2.671	2.525	2.624	2.498	2.391	2.381	2.294
(n+6)th Θ $\hbar/2$ (Js)/exp.(−34)	21.253	31.436	28.461	30.267	28.007	25.997	26.379	24.923	23.757
Zth Θ $\hbar/2$ (Js)/exp.(−34)	0.167	0.159	0.152	0.146	0.141	0.136	0.132	0.128	0.124
Heavy atoms	-	-	Ba	I	Mo	Rb	Br	Fe	Cu
1st Θ $\hbar/2$ (Js)/exp.(−34)	-	-	9.849	6.296	6.925	7.915	4.365	4.830	4.857
2nd Θ $\hbar/2$ (Js)/exp.(−34)	-	-	8.364	5.411	4.040	3.536	3.756	4.037	2.478

"Θ" can either be > or < one as the case may be and differs for different elements due to differences in ionization energies, and principal quantum numbers; it must also be positive.

The possibility of fractional energy levels seemed to have been recognized [20]. To this end, the following equation, derived as follows, can enable its determination given any reduced Bohr's radius. The equations in the literature [4] and the Coulomb equation are used for the derivation.

$$E_{\mathrm{i}(++)} = \frac{n_f^2 h^2}{8\pi^2 a_r^2 m_e} = \frac{e^2}{8\pi\varepsilon_0 a_r}, \tag{48}$$

where a_r, n_f, and $E_{\mathrm{i}(++)}$ are the reduced Bohr's radius (which may be reduced by two or more folds), the fractional energy level, and high energy closer to the nucleus. The equation of n_f is:

$$n_{\mathrm{f}} = \left(\frac{m_e \pi a_{\mathrm{r}}}{\varepsilon_0}\right)^{\frac{1}{2}} e/h, \tag{49}$$

With values of n_f and substitution into any of the preceding relevant equations, the values of $(n^3)^{\frac{1}{2}}\,\hbar/2$ can be calculated; they could be less than, greater than, or approximately equal to PUCs. The calculated values of $(n^3)^{\frac{1}{2}}\,\hbar/2$ only tell one that the PUCs may be lower, higher, or approximately equal to $\hbar/2$. Table 2 illustrates this issue using a hypothetical approach using hydrogen. Regardless of the written form in the literature, the relationships are as follows:

$(\langle Q^2\rangle - \langle Q\rangle^2)^{\frac{1}{2}}(\langle P^2\rangle - \sigma\langle P\rangle^2)^{\frac{1}{2}}$ and $[(\sum Q_i^2 - (\sum Q)^2/n_{st})^{\frac{1}{2}}/n_{st}][(\sum Q_i^2 - (\sum Q)^2/n_{st})^{\frac{1}{2}}/n_{st}]$ give the same value of PUC (5.9395 exp. (− 40) J s) which is $\ll \hbar/2$. The values, Bohr's radii, and average ionization energies used for illustrations are displayed in Table 2. Furthermore, Table 2 illustrates clearly the assertion that the closer an electron is to the nucleus, the lower the uncertainties in position and momentum. It is between 0.0264 and 0.0053 PUCs, corresponding to between $a_0/20$ and $a_0/100$ for H; 0.0114 and 0.00227 for He; 0.0443 and 0.00886 for Li; and 0.02895 and 0.00579 for Be.

Table 2 Factional energy levels and the fractions of half of reduced Planck's constants

H	H	a_r (exp.(−11) m)	H	He	Li	Be
E_H (kJ/mol).	a_0 (exp.(−11) m	$a_0/20$	n_f=0.2236	n_f =0.1581	n_f =0.2582	n_f =0.2236
1313	5.2885		Θ=0.0264	Θ=0.0114	Θ=0.0443	Θ=0.02895
1314	5.28454	$a_0/40$	n_f =0.1581	n_f =0.112	n_f =0.1826	n_f =0.1581
1317	5.27251		Θ=0.0132	Θ=0.0083	Θ=0.0221	Θ=0.0145
1312	5.29261	$a_0/60$	n_f =0.1291	n_f =0.0913	n_f =0.1491	n_f =0.1291
1318	5.26851		Θ =0.0088	Θ=0.0038	Θ=0.0148	Θ=0.0097
1304	5.32508	$a_0/80$	n_f=0.112	n_f =0.0792	n_f =0.1291	n_f =0.1118
1309	5.30474		Θ =0.0064	Θ=0.0029	Θ=0.0111	Θ=0.00724
1308	5.30879	$a_0/100$	n_f=0.0999	n_f =0.0701	n_f =0.1155	n_f =0.09996
-	-		Θ =0.0053	Θ=0.00227	Θ=0.00886	Θ=0.00579

Θ, n_f, and a_r are the positive fraction (<1), fractional energy level (quantum number), and reduced Bohr's radius: NB: Hypothetical values were used for illustrations. $\hbar/2$=0.5270469322 exp. (−34) J s which is exclusively for hydrogen. The values of the fractions of half of the reduced Planck's constant are given as: $\Theta\ \hbar\ /2$.

The concept of relative uncertainty (the ratio of $\Theta\ \hbar\ /2/\ \hbar\ /2=\Theta$) introduced in this research is intended to mean the probability (or probability density, ψ^2) of locating the position and momentum without resorting to the popular Schrödinger wave function and associated mathematical formalism. As Table 3 shows, the ψ^2 values could be as high as between ≈ 20 and 99 for H; ≈ 46 and 232 for He; ≈ 12 and 59 for Li; and ≈18 and 91 for Be. These values have implications for the physical and chemical properties of elements that make them amenable to one form of application or another.

Table 3 Relative uncertainty illustration using some elements at an arbitrarily chosen reduced Bohr's radius for hydrogenic atoms

a_r (exp.(−11) m)	H	He	Li	Be
$a_0/20$	n_f =0.2236	n_f =0.1581	n_f =0.2582	n_f =0.2236
	ψ^2=19.964	ψ^2=46.232	ψ^2=11.897	ψ^2=18.205
$a_0/40$	n_f =0.1581	n_f =0.112	n_f =0.1826	n_f =0.1581
	ψ2=39.928	ψ^2=63.500	ψ^2=23.848	ψ^2=36.348
$a_0/60$	n_f =0.1291	n_f =0.0913	n_f =0.1491	n_f =0.1291
	ψ^2=59.892	ψ^2=138.697	ψ^2=35.611	ψ^2=0.54.335
$a_0/80$	n_f =0.112	n_f =0.0792	n_f =0.1291	n_f =0.1118
	ψ^2=82.417	ψ^2=181.740	ψ^2=47.482	ψ^2=72.797
$a_0/100$	n_f =0.0999	n_f =0.0701	n_f =0.1155	n_f =0.09996
	ψ^2=99.443	ψ^2=232.179	ψ^2=59.497	ψ^2=91.027

ψ^2 is the relative uncertainty, otherwise christened "probability," of locating the position of an electron relative to the nucleus and its momentum, precluding the wave notion for the purpose of this research and for descriptive convenience.

There is a need to add that, the expansion of metals in hot weather is a result of the expansion of the 3-dimensional space available to them in the metallic lattice. The opposite is true upon exposure to cold weather. The free electrons become very mobile in hot weather, such that the uncertainties of the free electrons become very high. Indigestible food substances, like resistance starch, could find an explanation in the fact that where the molecular orbital electron cannot assume higher PUCs, the crystal structure may defy liquefaction; the electrons may be closer to the nucleus.

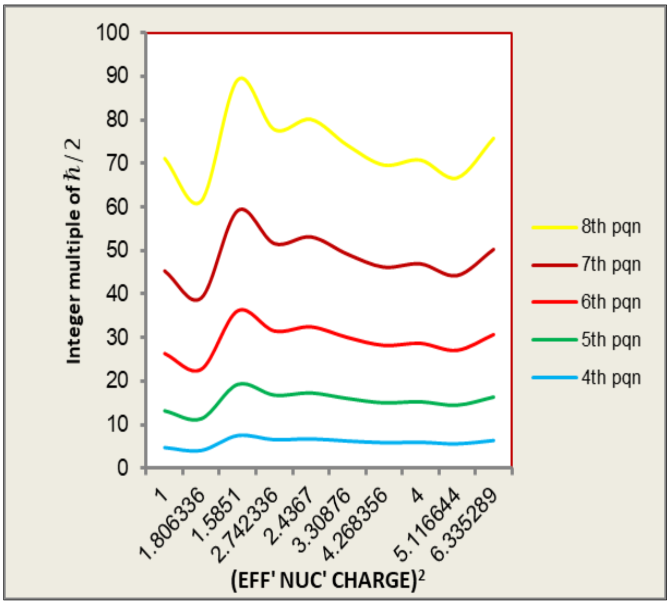

Figure 1 Variation of the minimum product of uncertainties (MPUCs, $\Theta\ \hbar/2$) versus the square of the effective nuclear charge. The different values of MPUCs were calculated for each element, from hydrogen to neon, at a given principal quantum number, pqn; EFF' and NUC' denote effective and nuclear, respectively. The effective nuclear charge is determined as described in the literature [21]

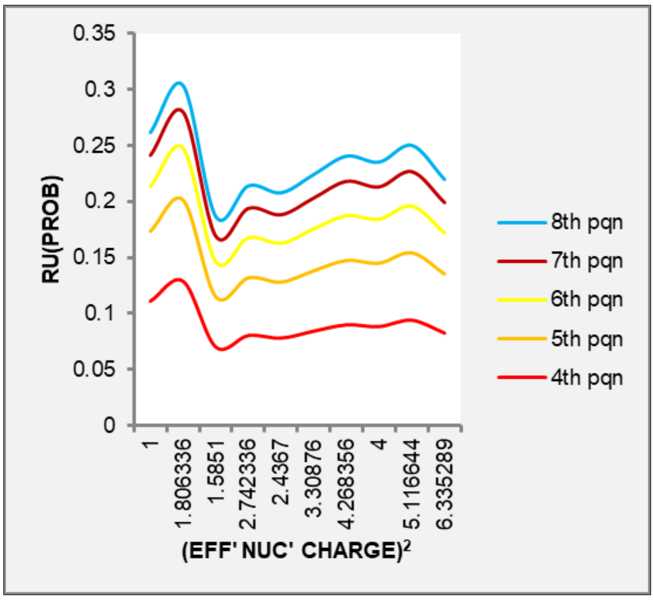

Figure 2 The variation of relative uncertainty otherwise christened probability with square of the effective nuclear charge (EFF' NUC' Charge). Prob. and RU stand for probability and relative uncertainty respectively both of which are symbolized as: (ψ^2)

With different elements, the minimum product of uncertainties (MPUCs i.e. $\Theta\ \hbar/2$) shows trend consistent with the variation in different ionization energies within the same period (Figure 1). The fall of the curve indicates a decrease in $\Theta\ \hbar/2$ while a rise indicates higher $\Theta\ \hbar/2$ values in line with lower ionization energy. It cannot be overemphasized to opine that any atom whose outermost shell electron has a high MPUCs, or $\Theta\hbar/2$, must also display a low probability

(ψ^2) of being located with its momentum precisely; the tendency to a precise position and momentum measurement is enhanced given a very low $\Theta\ \hbar/2$, with a concomitant high value of ψ^2. This is illustrated in Figure 2. The sharp rises in ψ^2 correspond to low values of $\Theta\ \hbar/2$

Values > $\hbar/2$ are shown by multielectron higher energy level atoms for clear reasons. It is obvious that larger atoms present higher uncertainty with respect to the outermost orbital or valence electron (s) due to the fact that the nuclear attraction for the electrons is weaker than for the inner electrons, which have much less freedom of motion. If n and E_i, respectively, are 1 and E_H (average ionization energy of hydrogen), Eq. (44) reproduces half of reduced Planck's constant. The fact that values >$\hbar$ /2 exist for some atoms means that the uncertainties for position and momentum for such atoms are very high.

Schrödinger quantum mechanics (SQM) or the Schrödinger equation (SE) has been described as one that successfully explains the hydrogen atomic orbital and residence of a single electron but fails where a multi-electron atom is the case. One may wish to comment that reference to atoms is closer to physical reality than recent experimental work [9], whose special method does not give any impression that either hydrogenic atoms or nonhydrogenic atoms or both can be referred to. This is unlike this study, in which, in addition, SE is inconsequential.

In his 1927 paper entitled (in Germany), Über den anschaulichen Inhalt der quantentheoretischen kinematic und Mecanik [7], the translation being the physical content of quantum kinematics and mechanics, names such as Bohr and Planck were stated; if one is not mistaken, Schrödinger was stated as to imply that quantum (or wave) mechanics and associated SE were mentioned text-wise. But SE is known to fail where multi-electron atoms are of interest. There is no ambiguity in the original Heisenberg equation, though the stereotypical limits on the upper and lower values of the product of two uncertainties have been refuted [9, 22]. Be that as it may, the research has come to the realization that these limits are not necessary if the energy levels of real atoms of elements as well as the kinetic energy of any electron at any energy level are considered.

The application of knowledge can drive the desire for the acquisition of more knowledge. The transfer of electrons in respiration and photosynthesis, digestion of resistant starch, expansion of metals, and the reactivity of the pi covalent bond electrons may have interpretations based on HUP. In this study, flexible limits are conditionally recognized in that the electron in an atom in its ground state presents a different physical characteristic from the same atom in an excited state. Hence, in stating the different inequalities, one takes into account possibilities such as an electron descending to a lower energy level, an electron transiently taking up positions between energy levels, and an electron undergoing partial expulsion, total expulsion, or ionization. This justifies Eqs (35a) to Eq. (38) with an increasing supporting view based on the experimental finding of Mills [20].

5. Conclusion

The equations for the determination of both fractions and multiples of $\hbar/2$ were derived. The fit was accomplished by relating Planck's constant to atomic properties. Such properties are average ionization energies and principal quantum numbers. The research shows that $\hbar/2$ is strictly for hydrogen. Higher multiples of $\hbar/2$ show that the uncertainties (position (a) and momentum (p)) could be very high, and vice versa. Fractions of and multiples of $\hbar/2$ have applications in the elucidation of the chemical and physical properties of matter as well as in biological processes.

Compliance with ethical standards

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Author contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Dedication

I dedicate this article to Mallam Aminu Kano in his honor. Amino Kano was a radical and progressive politician who was only ten years older than my late father, Mr. Festus Ola Udema (a teacher like Kano). His leftist (socialist) interest was solely focused on raising everyone's material, moral, educational, and human rights standards without harboring resentment, with the idea that this must be done whether or not he succeeds in politics.

Disclosure of conflict of interest

There is no competing interest; no financial interest with any government or corporate body or any individual.

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